AD-786 355

COLLAPSE OF PARTIALLY MIXED REGIONS IN STRATIFIED FLUIDS

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Prepared for:

Advanced Research Projects Agency

July 1974

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NRL Memorandum Report 2841	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER AD 786 355
4. TITLE (and Subtitle) COLLAPSE OF PARTIALLY MIXED REGIONS IN STRATIFIED FLUIDS		5. TYPE OF REPORT & PERIOD COVERED Interim report.
		6. PERFORMING ORG. REPORT NUMBER ARPA Order No. 2152
7. AUTHOR(s) J. P. Dugan, A. C. Warn-Varnas, and S. A. Piacsek		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D.C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Prob. 77-K03-54, 77-G01-11 Proj. No. WF-11-125-703
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Naval Air Systems Command Washington, D.C. 20360 (See reverse side)		12. REPORT DATE July 1974
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Stratified fluids
Internal waves
Density currents
Numerical methods
Numerical hydrodynamics
Navier-Stokes equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The collapse of a partially mixed fluid mass immersed in a stably stratified fluid is studied with a finite difference formulation of the Navier-Stokes equations. In a series of tests, the mixed region is assumed to have various types of density structure including ones derived from both uniform and nonuniform amounts of mixing. The shape of the mixed region is followed in detail. The temporal history of the width of the mixed region agrees with a simple analytical model in the

(Abstract continues)

20. (Continued abstract)

initial stage of collapse in the case where the amount of mixing is uniform. It also compares well with a previous analytical theory in the case of a small amount of mixing where that theory is valid.

Most of the energy stored in the mixed region is given up to the surrounding fluid in one Brunt-Vaisala period.

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COLLAPSE OF PARTIALLY MIXED REGIONS IN STRATIFIED FLUIDS

Introduction

There is a sizeable literature devoted to the collapse of regions of mixed fluid immersed in density stratified media. The physical application is the description of energy exchange between the stratified fluid and the region of mixed fluid. The mixed region is generated by an overturning internal wave (Long [1]) or by the turbulence in the wake of a moving obstacle (Schooley and Stewart [2], Schooley [3]). Thus, the generation of the mixed region is a turbulent process and the mixing is accomplished by the turbulent eddies. The mixed region grows in size by entrainment of surrounding fluid until the kinetic energy in the turbulence is dissipated. The mixed region is statically unstable and, when the turbulence decays sufficiently, it proceeds to collapse to its level of equilibrium.

There have been numerous attempts to model this collapse process. Except for models put forward by Ko [4] and Bergin [5], the collapse stage has been completely decoupled from the mixed region growth stage. Most physical models have taken as an initial condition a two-dimensional circular region of quiescent mixed fluid. The physical models have been represented mathematically in numerous ways to describe the ensuing collapse process. Schooley and Stewart [2] considered the mixed region to be a small perturbation from the ambient equilibrium state and solved the linearized, inviscid equations governing the resulting fluid motions by an eigenfunction expansion. The mixed region was assumed to be circular in shape with a density perturbation of Gaussian form. This method has been applied again by Schooley and Hughes [6] but the mixed region was assumed to be fully mixed in this case. In either case, the linearized governing equations are not strictly valid unless the density perturbation is indeed a very small one. The case in which the amount of mixing is small but uniform across the mixed region can be validly treated in this manner and it has been solved by Hartman and Lewis [7].

The fully mixed case is not as interesting physically as the original diffusely mixed case modelled by Schooley and Stewart [2] because of the sharp discontinuity in density at the edges of the mixed region. On the other hand, this discontinuity is helpful in distinguishing the fluid interior to the mixed region from that

Manuscript submitted June 25, 1974.

exterior to the region. The experiment of Wu [8] took advantage of this discontinuity by tagging the mixed region fluid with dye. This idealization has been helpful also in various mathematical models of the process. Mei [9], Padmanabhan et al [10], and Bell and Dugan [11] have solved models that account for nonlinearity in the governing equations but which omit realistic interactions of the mixed region with its surroundings. Wessel [12] and, more recently, Dugan et al [13] and Young and Hirt [14] have solved the full nonlinear equations numerically for the case of sharp discontinuity in the density at the edge of the mixed region. The numerical method is the most complete and accurate method if proper care is taken to verify the accuracy of the calculations. Thus, it appears to be the most useful method for any further studies of mixed region collapse.

The studies referred to above have brought to light most of the physical mechanisms present in the collapse phenomenon. This understanding is quite limited though in that knowledge is sparse for the more realistic case in which the turbulence does not completely homogenize the fluid in the mixed region. In particular, what are the mixed region shapes and energy balances when the mixing is uniform but not complete and when the mixing is not uniform. These questions are taken up here and the numerical method utilized by Dugan et al [13], and modified as indicated in the appendix, is helpful in answering them.

Analysis

The mixed region is taken to be circular in shape and to be at rest initially. The density structure is linearly increasing with depth outside the region so the Brunt-Vaisala frequency there is constant to the accuracy of the Boussinesq approximation. In the mixed region, the density profile is taken to have several forms as shown in Figure 1.

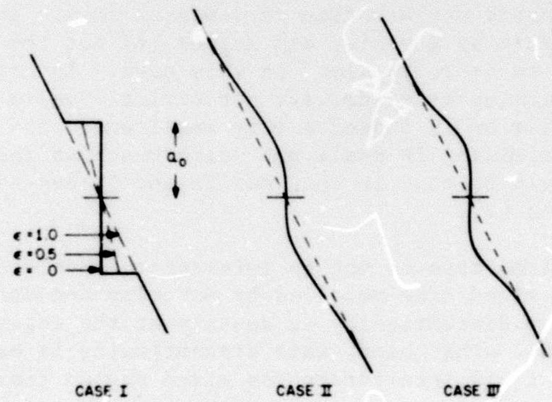


Figure 1

The case I is assumed to be uniformly mixed in that the density gradient in the interior is constant but is less than that outside the region. That is, the density profile is

$$\bar{\rho}(z) = \begin{cases} 1 - \beta z & r > a_0 \\ 1 - \beta_i z & r < a_0 \end{cases} \tag{1}$$

where r is the radial distance from the center of the mixed region. The case II has the density profile

$$\bar{\rho}(z) = 1 - \beta z \left\{ 1 - \exp(-r^2/a_0^2) \right\} \tag{2}$$

and the case III has

$$\bar{\rho}(z) = 1 - \beta z \left\{ 1 - \exp(-r^4/a_0^4) \right\} . \tag{3}$$

The latter case in a sense is intermediate between the case I having a sharp edge and the case II having a very diffuse edge. In the nonuniformly mixed cases II and III, the density perturbation is continuous and the mixing is complete only at the mixed region center. The amount of potential energy stored in the mixed region varies with the type of mixing and it is

$$\begin{aligned} &\text{I. } PE = \pi/8\, \rho_0 N^2 (1-\epsilon)^2 a_0^4 \\ &\text{II. } PE = \pi/16\, \rho_0 N^2 a_0^4 \\ &\text{III. } PE = \pi/16\, \rho_0 N^2 a_0^4 . \end{aligned} \tag{4}$$

In each case, the potential energy is defined as the work required to construct the mixed region from the linearly stratified surroundings in a reversible manner. The cases II and III have the same amount of potential energy and that amount is one half that of a fully mixed region of the same 'radius'.

The governing equations are the Navier-Stokes equations and, in the absence of turbulence, the nonlinear effects are important but the viscous and diffusive ones are not for reasonable scales. Several finite difference methods of solving the nonlinear equations are available. Wessel [12] initially solved a problem of this type in which the mixing was complete with a scheme similar to the one of Dugan et al [13]. The solutions obtained were in good agreement with the experiment of Wu [8] but the energy conservation properties of the code were uncertain. Young and Hirt [14] solved the same problem but the method used appears to dissipate the available energy quite rapidly. The modified MAC code used therein is stable only because of the presence of nonzero viscosity and the temporal history of the mixed region width does not compare well with the experiment. The method used by Dugan et al [13] has been shown to conserve energy quite well so it is a useful one to study the energetics in the problem at hand. It was also shown to predict results very similar to those of the experiment of Wu [8] for the fully mixed case. The numerical method is a modified form of that originally put forward by Williams [15]; the formulation is laid out in Dugan et al [13] and it has been modified as indicated in the appendix to allow grid stretching and to better conserve energy.

Uniform Mixing

The case I of partial but uniform mixing is considered first. The numerical solutions for the mixed region shape versus time are shown in Figure 2 for ϵ = .81 and .49 where

$$\epsilon = \beta_i \beta^{-1} \qquad (5)$$

and β_i and β are defined in expression (1). The solution for ϵ = .49 shows little change from the fully mixed case in the early stage of motion. However, eventually, all the shapes for even those values

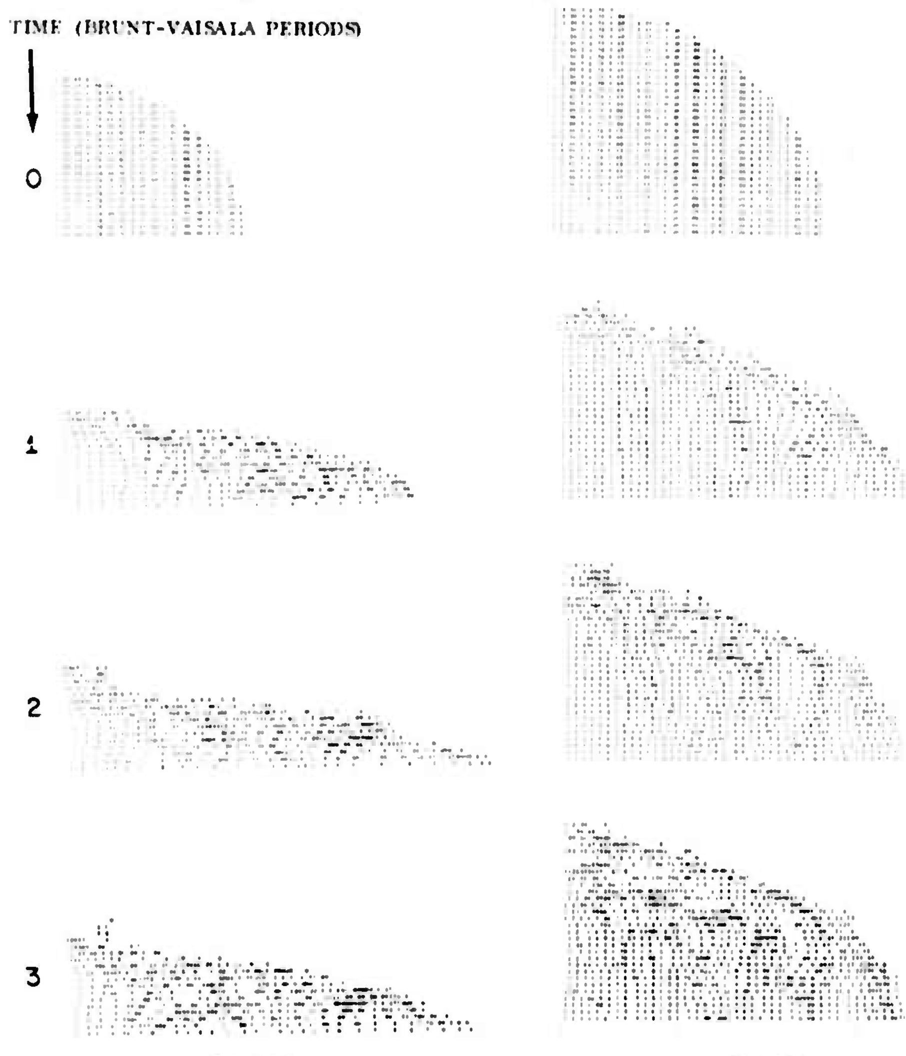


Figure 2

of ϵ approaching zero are different from the fully mixed case. The equilibrium positions of all fluid particles in the partially mixed region are above or below the mixed region centerline so that the shape never collapses to a line on the axis of the cylinder as it does in the fully mixed case. The final equilibrium shape does not appear to be far from an ellipse in cases for which $\epsilon > .5$.

The width of the mixed region as a function of time is shown in Figure 3 for several values of ϵ .

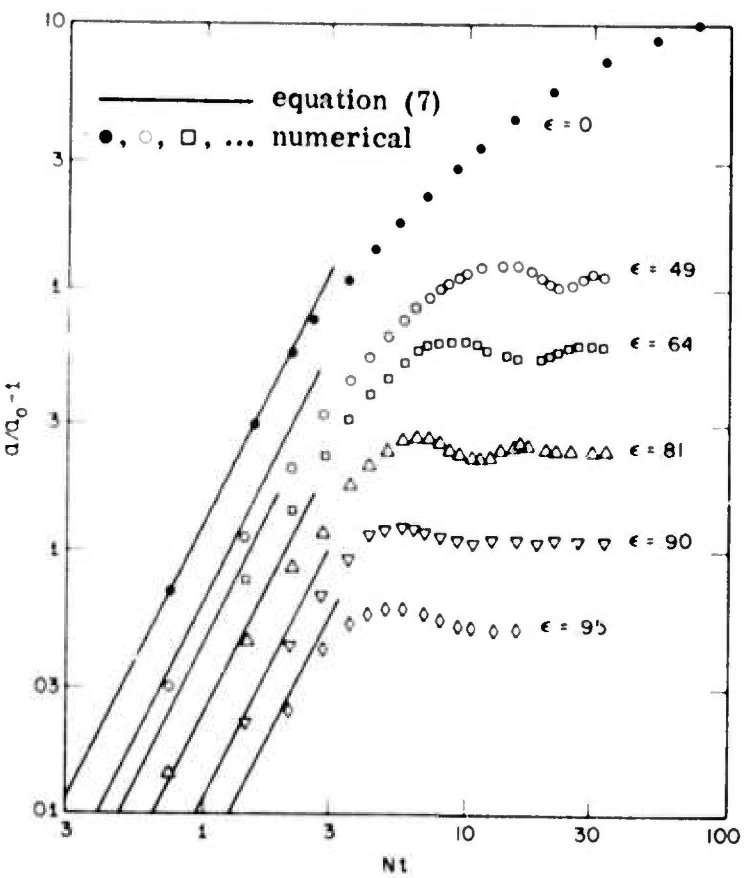


Figure 3

In all cases, the width increases to a maximum and exhibits a slightly underdamped oscillation to its equilibrium width.

This case of uniform, partial mixing is simple enough to be amenable to analysis. Hartman and Lewis [7] have obtained an analytical solution for this case and, to the validity of their solution, the width of the mixed region is easily shown to be

$$a(t)/a_0 = 2 - \epsilon - 2(1-\epsilon)\, J_1(Nt)\,(Nt)^{-1} \quad (6)$$

This solution is valid only when the amount of mixing is small; that is, it is valid in the limit of $\epsilon \to 1$. However, it is plotted in Figure 4 and it compares well with the numerical results for short time for all cases in which computations were made. The nonlinear

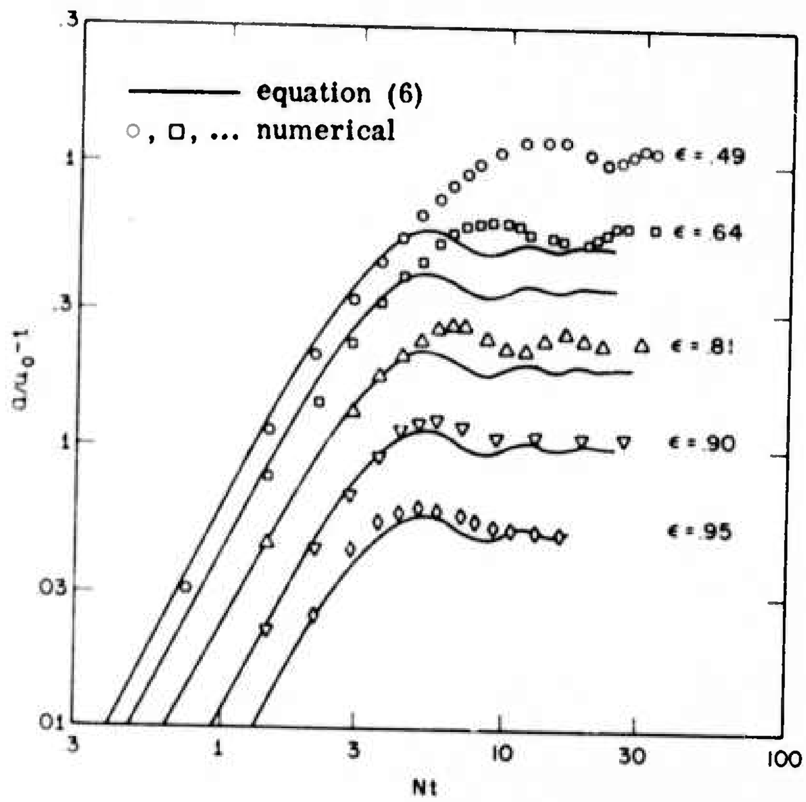


Figure 4

effects are clearly apparent in the mixed region widths in Figure 4 in the cases in which the mixing is greater than 20% ($\epsilon < .8$) as the numerical and the linear solutions digress. Even in the case of $\epsilon = .95$, though, the solutions are not identical. The nonlinear solutions outdistance the linear ones and they do not oscillate as quickly.

The solution method of Bell and Dugan [11] in which the energy in the mixed region is assumed to be conserved can be extended to apply to the case of partial mixing. The governing equation that corresponds to equation (12) in Bell and Dugan [11] is

$$(1+\alpha^{-4})\dot{\alpha}^2 + (1-2\epsilon\alpha)\alpha^{-2} - (1-2\epsilon) = 0 .$$

The solution of this equation for small time goes like

$$a(t)/a_0 \approx 1 + \frac{1-\epsilon}{4} N^2 t^2 - O(N^4 t^4) .$$

Of course this expression is never strictly valid since the mixed region never conserves its energy. On the other hand, it is not restricted to $\epsilon \approx 1$. Dugan et al [13] have modified this analysis with the assumption that there is an equipartition of kinetic energy inside and outside the mixed region and have verified the resulting analytical prediction with the numerical solutions in the case of complete mixing. Making that assumption here, the governing equation is

$$2(1+\alpha^{-4})\dot{\alpha}^2 + (1-2\epsilon\alpha)\alpha^{-2} - (1-2\epsilon) = 0.$$

This cannot be solved in closed form as in the fully mixed case but the solution valid for small time is

$$a(t)/a_0 \approx 1 + \frac{1-\epsilon}{8} N^2 t^2 - O(N^4 t^4). \tag{7}$$

It is of interest to note that the short time behavior of the solution of Hartman and Lewis [7] (see equation (6) above) is identical to this expression. The comparison between the numerical results and the asymptotic solution (7) above is shown in Figure 3.

The energy content of the mixed region can be obtained approximately from the foregoing results. Bell and Dugan [11] were able to compute energies under the assumption that the mixed region shape was elliptic. Dugan et al [13] showed that the region shape was not elliptic after about one half of a Brunt-Vaisala period in the fully mixed case. However, the energetics are integrals of the solution so they tend to be only weakly dependent upon the details of the mixed region shape. The validity of the assumption of an elliptic shape was shown by Dugan et al [13] by computing the energetics numerically for times up to about one Brunt-Vaisala period. The results agreed reasonably well with those predicted by Bell and Dugan [11]. The mixed region shapes predicted numerically here and shown in Figure 2 are even more elliptic than in the fully mixed case so the assumption of an elliptic shape for energetics computations is a better one here.

From the above discussion, it is apparent that the energy content of the mixed region can be computed from its width. For an elliptic shape in the uniform, partially mixed case, the potential energy is

$$PE = \pi/8\, \rho_0 N^2 (1-\epsilon a a_0^{-1})^2 a_0^6 a^{-2}, \tag{8a}$$

valid to the Boussinesq approximation, and the kinetic energy is

$$KE = \pi/8\, \rho_0 \left(\frac{da}{dt}\right)^2 a_0^2 (1 + a_0^4 a^{-4}). \tag{8b}$$

The values of $a(t)$ and da/dt can be obtained from Figure 3 to give the results shown in Figure 5. The case of $\epsilon = .81$ is illustrated.

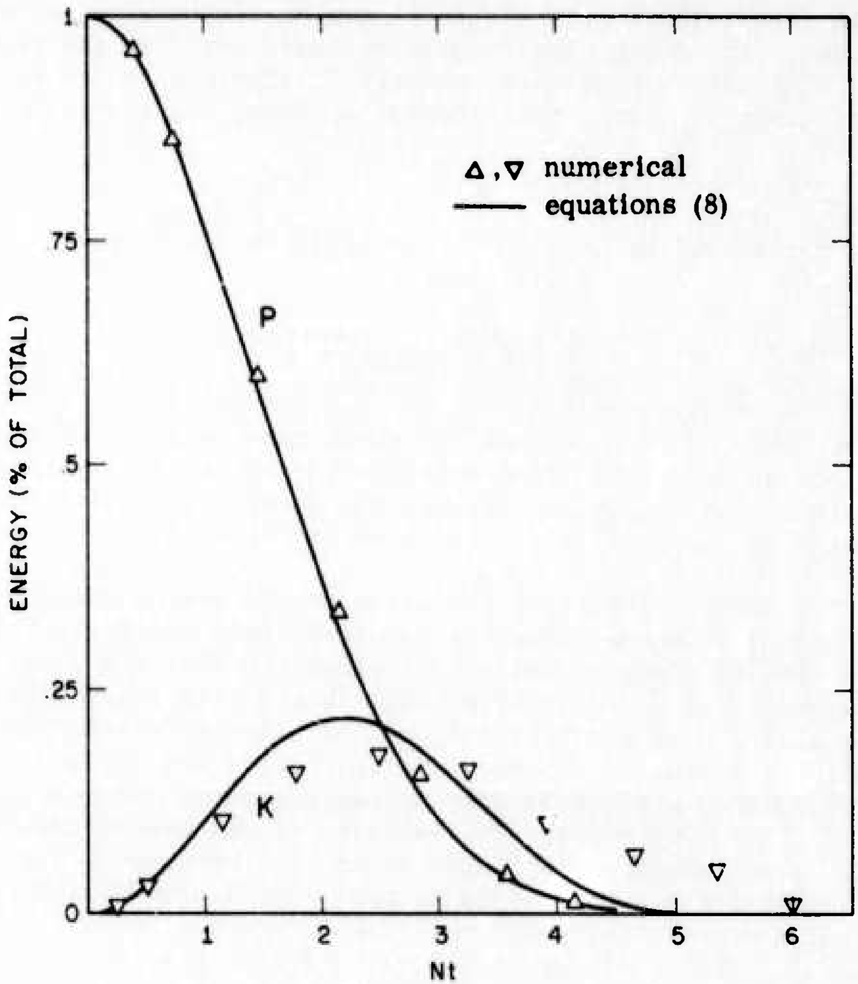


Figure 5

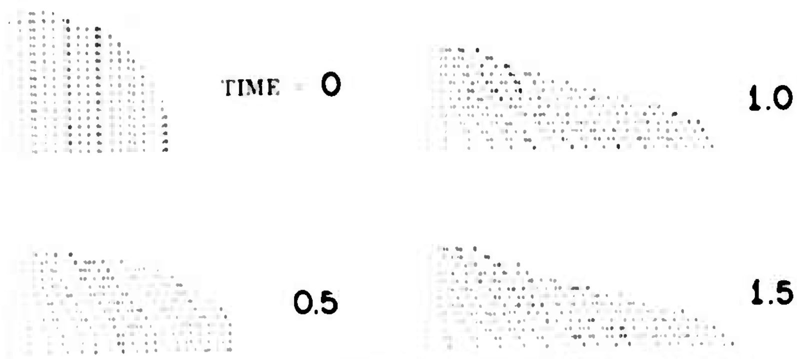
The result is sufficient to conclude that, as in the fully mixed case, the energy content of the mixed region is largely given up to the exterior fluid in one Brunt-Vaisala period. This case of $\epsilon = .81$ is quite representative of the other cases run so plots of those results are omitted.

Figure 5 also shows the kinetic and potential energy content of the mixed region as predicted by equations (8a,8b) in which $a(t)$ and da/dt are given by the theory of Hartman and Lewis [7], i.e. expression (6) here. Although the result for the kinetic energy is somewhat different and the linear theory strictly is not valid in this case, the results predicted by the linear theory support the numerical prediction of the quick release of energy by the mixed region.

Gaussian Mixing

The density perturbation of Gaussian form in case II is quite diffuse compared with the case I. Analyses of mixed region shapes and widths are not very meaningful since the edge of the mixed region is not clearly defined. Nevertheless, approximations to shapes can be determined if one is content to follow fluid particles that have some initially given positions.

The shape of a dense mass of particles that is initially in a circle having the radius a_0 is plotted with time in Figure 7. Compared with the previous case, there is an interesting difference



TIME IN BRUNT-VAISALA PERIODS

Figure 7

in the final equilibrium positions of the particles. The shape is not elliptic at all but it shows a distinct peak near the original vertical centerline. This equilibrium shape is explained by the type of density perturbation. Since the amount of mixing goes to zero smoothly near the edge of the mixed region, the final equilibrium positions of the particles near the top and the bottom of the region are not very far from the initial positions. Thus, although the particles near the center move appreciably because of the greater amount of mixing there, those near the upper and lower edge move only slightly.

The energy content of the mixed region is not clearly defined since there is no 'edge' to the mixed region so no computations are made.

Modified Gaussian Mixing

As in the previous case, numerical results have been obtained for the case III. The particle distribution that was initially in the region $r \leq a_0$ is displayed versus time in Figure 8. This initial density perturbation is closer to the case I having a sharp edge but the small amount of mixing near the edge in this case continues to give final equilibrium shapes with a peak near the vertical centerline.

Finally, one more observation about the final width of the mixed region is useful. The potential energy for a mixed region of elliptic shape is given by expression (8a). Since, in the final stage of collapse, the potential energy is all given up to the exterior fluid, this expression implies that the final equilibrium half-width of the mixed region is

$$a = a_o \epsilon^{-1} . \tag{9}$$

This expression is in good agreement with the numerical results as shown in Figure 6.

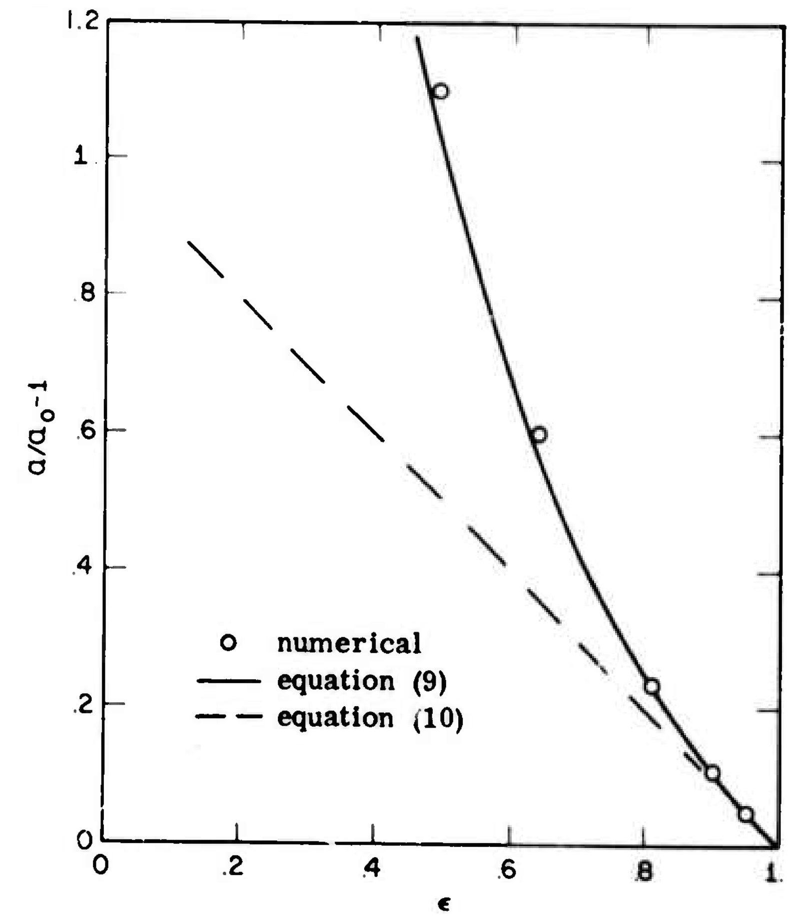


Figure 6

The result is compared with that of Hartman and Lewis [7] which implies that

$$a = a_o (2-\epsilon) \tag{10}$$

and which, evidently, is valid only in the limit of very small mixing.

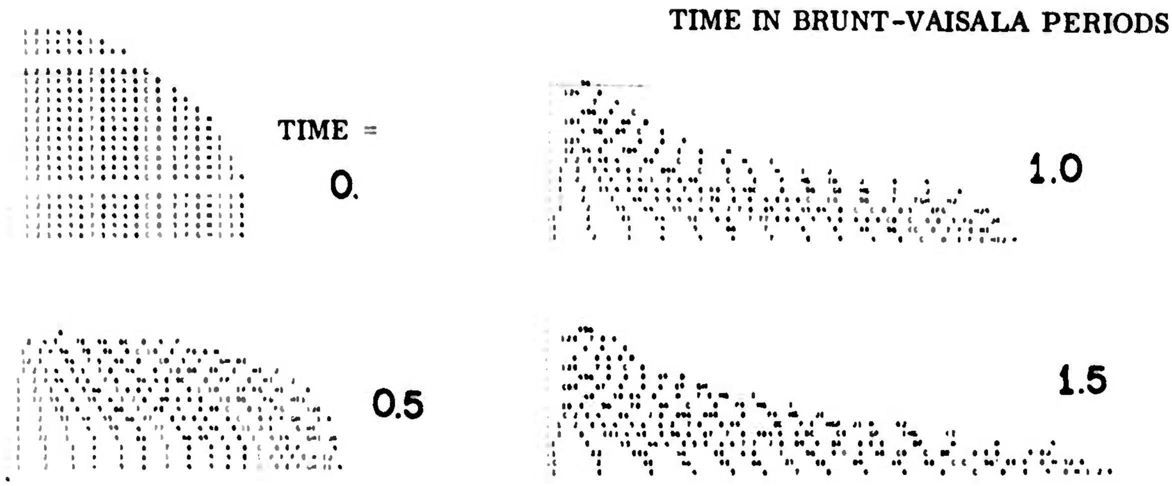


Figure 8

Conclusions

The final equilibrium shape of partially mixed regions is shown to depend crucially upon the initial density perturbation that defines the mixed region. The equilibrium shapes obtained for uniformly mixed regions are elliptic in cross section but the shapes obtained in cases of nonuniformly mixed regions show a distinct peak near the vertical centerline.

The case of uniform, partial mixing is less realistic than the nonuniformly mixed cases but it lends itself to energetics computations. In this case, for any degree of mixing, the energy content of the mixed region is practically all given up to the exterior fluid by the time of one Brunt-Vaisala period after the initiation of collapse. There is little reason to believe that this result would be any different for other types of mixing. Also, the final equilibrium width of the mixed region is close to the width predicted by a simple theory; that is

$$\lim_{t \to \infty} a(t) \sim a_0 N_0^2 N_i^{-2},$$

where N_0 is the Brunt-Vaisala frequency outside the region and N_i is the initial uniform Brunt-Vaisala frequency inside the region.

The temporal dependence of the mixed region width in the case of uniform, partial mixing is compared with analytic results from an integral energetics model and a linearized differential model. Non-linear effects are shown to be important in mixed regions that are mixed as much as 20% or more; that is, when $N_i^2 N_0^{-2} \lesssim .8$.

References

1. Long, R.R.: A theory of turbulence in stratified fluids. J. Fluid Mech. 42 (1970), 349-365.

2. Schooley, A.H. and R.W. Stewart: Experiments with a self-propelled body submerged in a fluid with a vertical density gradient. J. Fluid Mech. 15 (1963), 83-96.

3. Schooley, A.H.: Wake collapse in a stratified fluid. Science 157 (1967), 421-423.

4. Ko, D.R.S.: Collapse of a turbulent wake in a stratified medium. Fleet Studies Final R port, TRW, Oct., 1971.

5. Bergin, J.M.: Internal wave generation caused by the growth and collapse of a mixed region. NRL Report 7568, June, 1973.

6. Schooley, A.H. and B.A. Hughes: An experimental and theoretical study of internal waves generated by the collapse of a two-dimensional mixed region in a density gradient. J. Fluid Mech. 51 (1972), 159-175.

7. Hartman, R.J. and H.W. Lewis: Wake collapse in a stratified fluid: linear treatment. J. Fluid Mech. 51 (1972), 613-618.

8. Wu, J.: Mixed region collapse with internal wave generation in a density-stratified medium. J. Fluid Mech. 35 (1969), 531-544.

9. Mei, C.C.: Collapse of a homogeneous fluid mass in a stratified fluid. Proc. 12th Intl. Cong. Appl. Mech., Aug., 1968, Stanford, 321-330.

10. Padmanabhan, H., W.F. Ames, J.F. Kennedy, and T.K. Hung: A numerical investigation of wake deformation in density stratified fluids. J. Engrg. Maths. 4 (1970), 229-241.

11. Bell, T.H., Jr. and J.P. Dugan: Mixed region collapse in a stratified fluid. NRL Memorandum Report 2564, March, 1973.

12. Wessel, W.R.: Numerical study of the collapse of a perturbation in an infinite density stratified fluid. Phys. Fluids, Supplement II (1969), 171-176.

13. Dugan, J.P., A.C. Warn-Varnas, and S.A. Piacsek: Numerical model for mixed region collapse in a stratified fluid. NRL Memorandum Report 2597, June, 1973.

14. Young, J.A. and C.W. Hirt: Numerical calculation of internal wave motions. J. Fluid Mech. 56 (1972), 265-276.

15. Williams, G.P.: Numerical investigation of the three-dimensional Navier-Stokes equations for incompressible flow. J. Fluid Mech. 37 (1969), 727-750.

16. Piacsek, S.A. and G.P. Williams: Conservation properties of convection difference schemes. J. of Computational Physics 6 (1970), 392-405.

17. Bryan, K.: A scheme for numerical integration of the equations of motion on an irregular grid free of nonlinear instability. Monthly Weather Review 94 (1966), 39-40.

Appendix I

The purpose of this section is to give details on the additional numerical features introduced by the stretching of the mesh employed in Dugan et al [13] in both the x- and z- directions. The reader is referred to section 2 of the above reference to obtain a complete description of the conservation, accuracy and stability of the constant mesh scheme; a discussion is also contained in Williams [15] and Piacsek and Williams [16].

Figure 9 illustrates the stretched mesh in one direction and the arrangement of the variables on their respective grid points. A discussion of the general approach to conservation on stretched

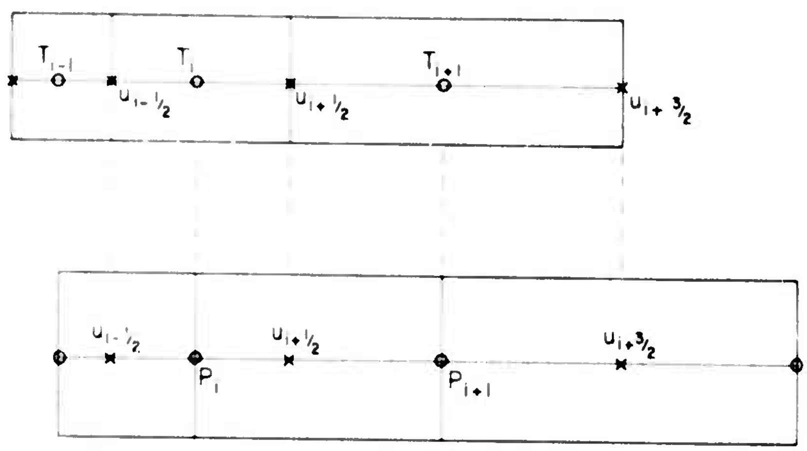


Figure 9

meshes is given by Bryan [17] and will not be repeated here. However, we will illustrate below the essential ideas by applying the scheme to the one-dimensional transport of momentum and temperature. For inviscid flows, the equations

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x} \qquad (I.1)$$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} \qquad (I.2)$$

will serve adequately to demonstrate the technique. Using Figure 9 and the quadratically conserving scheme of Piacsek and Williams [16], we may write

$$\left(\frac{\partial u}{\partial t}\right)_{i+1/2} = -\frac{u_{i+1}u_{i+3/2} - u_i u_{i-1/2}}{\Delta x_{i+1/2}} - \frac{p_{i+1} - p_i}{\Delta x_{i+1/2}}$$

$$\left(\frac{\partial T}{\partial t}\right)_i = -\frac{u_{i+1/2}T_{i+1} - u_{i-1/2}T_{i-1}}{\Delta x_i} \qquad (I.3)$$

Note from Figure 9 that $\Delta x_{i+1/2}$ and Δx_i refer to elements of different mesh sequences, one 'centered' on $u_{i+1/2}$ and the other on T_i, respectively. Upon multiplying expression (I.3) by $\Delta x_{i+1/2}$ and expression (I.4) by Δx_i and summing over i, the sums encountered in Dugan et al [13] are produced and these were shown to be conserved. However, whereas the T-boxes are actually centered on T_i, the u-boxes are centered on $u_{i+1/2}$, resulting in a loss of second order accuracy in the truncation error unless the mesh is stretched in certain special ways. The current mesh stretching has only first order accuracy, and is given by

$$\xi = \left(e^{x/a_0} - 1\right)\left(e^{x/a_0} + 1\right)^{-1} \tag{I.5}$$

In the calculations in this report, the mesh was stretched such that the final grid length in the horizontal was eighteen times the initial length, and the total box size was eight radii wide and four radii high. The stretching enabled a reduction in the number of horizontal mesh points by a factor of five.

In addition to the changes outlined above, the computations performed in solving the Poisson equation in Dugan et al [13] are now done in double precision so that roundoff errors are minimized. This gives an order of magnitude improvement in the conservation of total energy in the computation box; the change in total energy in fourteen Brunt-Vaisala periods of computation is less than .5% versus somewhat less than 5% in single precision arithmetic.